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## A Remark on Finiteness and Duality of D-Modules

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The purpose of this paper is to prove a theorem on finite dimensionality of the cohomology groups of analytic differential complexes on compact real manifolds. This generalizes the classical finiteness theorem for elliptic differential complexes.

In this paper, a manifold is always assumed to be paracompact. For sheaves and functors, we follow the notations of [KS]. For a sheaf  $\mathcal{F}$  on a topological manifold  $X$ ,  $\Gamma(\mathcal{F})$  denotes the set of global sections of  $\mathcal{F}$ , and  $\Gamma_c(\mathcal{F})$  of global sections with compact support.  $R\Gamma$  and  $R\Gamma_c$  denote their right derived functors.

### 1. Main Result

Let  $X$  be a complex manifold,  $n = \dim_{\mathbf{C}} X$ . Let  $\mathcal{O}$  denote the sheaf of holomorphic functions on  $X$ ,  $\Omega^n$  the sheaf of holomorphic  $n$ -forms on  $X$ , and  $\mathcal{D}$  the sheaf of rings of differential operators on  $X$  (of finite order).

Let  $T^*X$  denote the cotangent bundle of  $X$ .

For a coherent right  $\mathcal{D}$ -module  $\mathcal{M}$  on  $X$ ,  $\text{Ch}(\mathcal{M})$  denotes its characteristic variety;  $\text{Ch}(\mathcal{M})$  is a  $\mathbf{C}^\times$ -invariant closed analytic subset of  $T^*X$  and  $\dim(\text{Ch}(\mathcal{M})) \geq n$  (see [SKK]). Let  $\text{Mod}(\mathcal{D}^\circ)$  be the abelian category of right  $\mathcal{D}$ -modules,  $D^b(\mathcal{D}^\circ)$  its derived category with bounded cohomology. Let  $D_{g, \text{coh}}^b(\mathcal{D}^\circ)$  be the full triangulated subcategory of  $D^b(\mathcal{D}^\circ)$  consisting of bounded complexes with good coherent cohomology groups. [We say that a  $\mathcal{D}$ -module  $\mathcal{M}$  is good coherent if, for any relatively compact open subset  $U$  of  $X$ , there exists a finite filtration  $G$  of  $\mathcal{M}|_U$  by  $\mathcal{D}$ -modules such that

[Sai, 1.15].] For an object  $\mathcal{M}^\bullet$  of  $D_{g,\text{coh}}^b(\mathcal{D}^\circ)$ ,  $\text{Ch}(\mathcal{M}^\bullet)$  denotes the union of  $\text{Ch}(H^k \mathcal{M}^\bullet)$ ,  $k \in \mathbf{Z}$ .

Let  $D^b(X)$  denote the derived category with bounded cohomology of the abelian category of  $\mathbf{C}_X$ -modules,  $D_{\mathbf{R}\text{-c}}^b(X)$  the full triangulated subcategory of  $D^b(X)$  consisting of  $\mathbf{R}$ -constructible objects [KS, Sect.8.4]. For an object  $F$  of  $D^b(X)$ ,  $\text{SS}(F)$  denotes the micro-support of  $F$  [KS, Sect.5.1]. If  $F$  is  $\mathbf{R}$ -constructible,  $\text{SS}(F)$  is then an  $\mathbf{R}_+$ -invariant closed subanalytic subset of  $T^*X$ . (But we do not need this fact in this paper.)

For an object  $(\mathcal{M}^\bullet, F)$  of  $D_{g,\text{coh}}^b(\mathcal{D}^\circ) \times D_{\mathbf{R}\text{-c}}^b(X)$ ,  $\mathcal{M}^\bullet \otimes F$  denotes the tensor product over  $\mathbf{C}$  and is an object of  $D^b(\mathcal{D}^\circ)$ .

**Theorem 1.** *Let  $(\mathcal{M}^\bullet, F)$  be an object of  $D_{g,\text{coh}}^b(\mathcal{D}^\circ) \times D_{\mathbf{R}\text{-c}}^b(X)$ . Assume that, for any irreducible component  $V$  of  $\text{Ch}(\mathcal{M}^\bullet)$ ,  $V \cap \text{SS}(F)$  is contained in the zero section  $T_X^*X$  if  $\dim V \neq n$ . Suppose  $\text{Supp}(\mathcal{M}^\bullet) \cap \text{Supp}(F)$  is compact. Then every cohomology group of  $\text{R}\Gamma(\mathcal{M}^\bullet \otimes F \otimes_{\mathcal{D}}^L \mathcal{O})$  and  $\text{RHom}_{\mathcal{D}}(X; \mathcal{M}^\bullet \otimes F, \Omega^n)$  is finite dimensional and*

$$(1.0) \quad \text{RHom}_{\mathcal{D}}(X; \mathcal{M}^\bullet \otimes F, \Omega^n)[n] \longrightarrow \text{Hom}_{\mathbf{C}}(\text{R}\Gamma(\mathcal{M}^\bullet \otimes F \otimes_{\mathcal{D}}^L \mathcal{O}), \mathbf{C})$$

*is an isomorphism in  $D^b(\mathbf{C})$ . Hence, for any  $k \in \mathbf{Z}$ ,*

$$\text{Tor}_k^{\mathcal{D}}(\mathcal{M}^\bullet \otimes F, \mathcal{O}) \quad \text{and} \quad \text{Ext}_{\mathcal{D}}^{k+n}(X; \mathcal{M}^\bullet \otimes F, \Omega^n)$$

*are vector spaces of finite dimension and dual to each other.*

*Remark.* We say that  $(\mathcal{M}^\bullet, F)$  is an elliptic pair if  $\text{Ch}(\mathcal{M}^\bullet) \cap \text{SS}(F) \subset T_X^*X$  [SS]. In that case, Theorem 1 is proved in [SS]. On the other hand, if  $\mathcal{M}^\bullet$  is holonomic,  $(\mathcal{M}^\bullet, F)$  satisfies the hypothesis of Theorem 1 for any object  $F$  of  $D_{\mathbf{R}\text{-c}}^b(X)$ .

Let  $M$  be a real analytic manifold of dimension  $n$ ,  $X$  a complex neighborhood of  $M$ . Let  $T_M^*X$  denote the conormal bundle of  $M$ .  $\mathcal{A}_M$  denotes the sheaf of real analytic functions on  $M$ , and  $\mathcal{B}_M$  of hyperfunctions;  $\mathcal{A}_M$  and  $\mathcal{B}_M$  are  $\mathcal{D}|_M$ -modules. Let

$$\mathcal{B}_M^{(n)} = \mathcal{B}_M \otimes_{\mathcal{A}} (\Omega^n \otimes \text{or}_{M/X}),$$

where  $\text{or}_{M/X}$  is the relative orientation sheaf of  $M$  in  $X$ ;  $\mathcal{B}_M^{(n)}$  is a right  $\mathcal{D}|_M$ -module.

As an immediate corollary of Theorem 1, we have the following finiteness and duality theorem of analytic differential complexes on compact real manifolds.

**Corollary 2.** *Let  $M$  be a compact real analytic manifold of dimension  $n$ . Let  $\mathcal{M}^\bullet$  be an object of  $D_{g, \text{coh}}^b(\mathcal{D}^\circ)$ . Assume that, for any irreducible component  $V$  of  $\text{Ch}(\mathcal{M}^\bullet)$ ,  $V \cap T_M^*X$  is contained in the zero section if  $\dim V \neq n$ . Then, for any  $k \in \mathbf{Z}$ ,  $\text{Tor}_k^{\mathcal{D}}(\mathcal{M}^\bullet, \mathcal{A}_M)$  and  $\text{Ext}_{\mathcal{D}}^k(M; \mathcal{M}^\bullet, \mathcal{B}_M^{(n)})$  are vector spaces of finite dimension and dual to each other.*

Let  $E^k$ ,  $0 \leq k \leq k_0$ , be holomorphic vector bundles over  $X$  and let

$$(1.1) \quad \mathcal{O}(E^0) \xrightarrow{L_0} \mathcal{O}(E^1) \xrightarrow{L_1} \dots \longrightarrow \mathcal{O}(E^{k_0})$$

be a differential complex of vector bundles, where  $L_k$  is a differential operator mapping  $\Gamma(\mathcal{O}(E^k))$  to  $\Gamma(\mathcal{O}(E^{k+1}))$ . [For a holomorphic vector bundle  $E$ ,  $\mathcal{O}(E)$  denotes the sheaf of holomorphic sections of  $E$ .]

Let  $\mathcal{M}^k = \mathcal{O}(E^k) \otimes_{\mathcal{O}} \mathcal{D}$  and

$$(1.2) \quad \mathcal{M}^\bullet = \left[ 0 \longrightarrow \mathcal{M}^0 \xrightarrow{L_0} \mathcal{M}^1 \xrightarrow{L_1} \dots \longrightarrow \mathcal{M}^{k_0} \longrightarrow 0 \right],$$

where  $L_k$  acts on  $\mathcal{M}^k$  by left multiplication;  $\mathcal{M}^\bullet$  is then an object of  $D_{g, \text{coh}}^b(\mathcal{D}^\circ)$ . Then  $\text{R}\Gamma(\mathcal{M}^\bullet \otimes_{\mathcal{D}}^L \mathcal{A}_M)$  is represented by a differential complex

$$(1.3) \quad 0 \longrightarrow \Gamma(M, E^0) \xrightarrow{L_0} \Gamma(M, E^1) \xrightarrow{L_1} \dots \longrightarrow \Gamma(M, E^{k_0}) \longrightarrow 0$$

and  $\text{Tor}_{-k}^{\mathcal{D}}(\mathcal{M}^\bullet, \mathcal{A}_M)$  is its  $k$ -th cohomology group, where  $\Gamma(M, E^k)$  denotes the space of analytic sections of  $E^k$  on  $M$ . For a vector bundle  $E$ , let us set  $\mathcal{B}^{(n)}(E) = \mathcal{O}(E) \otimes_{\mathcal{O}} \mathcal{B}_M^{(n)}$ .  $\text{RHom}_{\mathcal{D}}(M; \mathcal{M}^\bullet, \mathcal{B}_M^{(n)})$  is represented by

$$0 \longleftarrow \Gamma(M, \mathcal{B}^{(n)}(E_0^*)) \xleftarrow{L_0} \Gamma(M, \mathcal{B}^{(n)}(E_1^*)) \xleftarrow{L_1} \dots \longleftarrow \Gamma(M, \mathcal{B}^{(n)}(E_{k_0}^*)) \longleftarrow 0,$$

where  $E_k^*$  is the dual bundle of  $E^k$  and  $L_k$  acts on  $\mathcal{B}^{(n)}(E_k^*)$  by right multiplication;  $\text{Ext}_{\mathcal{D}}^{-k}(M; \mathcal{M}^\bullet, \mathcal{B}_M^{(n)})$  is its  $k$ -th homology group. The pairing of  $\text{Tor}_{-k}^{\mathcal{D}}(\mathcal{M}^\bullet, \mathcal{A}_M)$  and  $\text{Ext}_{\mathcal{D}}^{-k}(M; \mathcal{M}^\bullet, \mathcal{B}_M^{(n)})$  is induced from

$$\Gamma(M, E^k) \times \Gamma(M, \mathcal{B}^{(n)}(E_k^*)) \rightarrow \mathbf{C}, \quad (u, v) \mapsto \int_M \langle u, v \rangle,$$

$\langle u, v \rangle$  being the pairing of  $E^k$  and  $E_k^*$ .

*Remark.* If (1.3) is an elliptic complex of vector bundles on  $M$ , for  $\mathcal{M}^\bullet$  given by (1.2),  $\text{Ch}(\mathcal{M}^\bullet) \cap T_M^*X$  is contained in the zero section. The converse is not true in general.

## 2. Proof of Theorem 1

We can assume that  $H^k \mathcal{M}^\bullet = 0$  for any  $k \neq 0$ ; in what follows,  $\mathcal{M}$  denotes a coherent right  $\mathcal{D}$ -module on  $X$ .

Let  $\mathcal{M}^* = \mathcal{E}xt_{\mathcal{D}}^n(\mathcal{M}, \mathcal{D})$ ; then  $\mathcal{M}^*$  is a holonomic left  $\mathcal{D}$ -module, and we have an injective  $\mathcal{D}$  homomorphism  $\mathcal{E}xt_{\mathcal{D}}^n(\mathcal{M}^*, \mathcal{D}) \rightarrow \mathcal{M}$ . Let  $\mathcal{M}^{**} = \mathcal{E}xt_{\mathcal{D}}^n(\mathcal{M}^*, \mathcal{D})$ , and  $\mathcal{N} = \mathcal{M}/\mathcal{M}^{**}$ ; then  $\mathcal{M}^{**}$  is a holonomic  $\mathcal{D}$ -module, and the sequence

$$(2.0) \quad 0 \rightarrow \mathcal{M}^{**} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0$$

is exact. Since  $\mathcal{E}xt_{\mathcal{D}}^{n-1}(\mathcal{M}^{**}, \mathcal{D}) = 0$  and  $\mathcal{E}xt_{\mathcal{D}}^n(\mathcal{M}, \mathcal{D}) \rightarrow \mathcal{E}xt_{\mathcal{D}}^n(\mathcal{M}^{**}, \mathcal{D})$  is an isomorphism, we see that  $\mathcal{E}xt_{\mathcal{D}}^n(\mathcal{N}, \mathcal{D}) = 0$ . Hence, by [K2, 2.11],  $\text{Ch}(\mathcal{N})$  has no irreducible components of codimension  $n$ . Since  $\text{Ch}(\mathcal{N}) \subset \text{Ch}(\mathcal{M})$ , by the hypothesis of the theorem,  $\text{Ch}(\mathcal{N}) \cap \text{SS}(F)$  is contained in the zero section; therefore  $(\mathcal{N}, F)$  is elliptic in the sense of [SS]. Moreover, by the definition of  $\mathcal{N}$ , if  $\mathcal{M}$  is a good coherent  $\mathcal{D}$ -module,  $\mathcal{N}$  is also good coherent.

Since Theorem 1 is proved for elliptic pairs in [SS, Part 1], by exact sequence (2.0), we may assume from the beginning  $\mathcal{M}$  to be holonomic. If  $\mathcal{M}$  is holonomic, by Kashiwara's theorem [K1],  $\mathcal{M} \otimes_{\mathcal{D}}^{\mathbf{L}} \mathcal{O}$  is  $\mathbf{C}$ -constructible. Hence  $(\mathcal{M} \otimes_{\mathcal{D}}^{\mathbf{L}} \mathcal{O}) \otimes F$  is an  $\mathbf{R}$ -constructible sheaf on  $X$ . Its support being compact by assumption, by [KS, Prop.8.4.8],  $H^k \text{R}\Gamma((\mathcal{M} \otimes_{\mathcal{D}}^{\mathbf{L}} \mathcal{O}) \otimes F)$  is finite dimensional for all  $k \in \mathbf{Z}$ . In the same way, the  $\mathbf{C}$ -constructibility of  $\text{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \Omega^n)$  yields the finite dimensionality of  $H^k \text{R}\Gamma(\text{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes F, \Omega^n))$ . This completes the proof of the finiteness part.

We now prove (1.0) to be an isomorphism for a holonomic  $\mathcal{D}$ -module  $\mathcal{M}$ , assuming  $\text{Supp}(\mathcal{M}) \cap \text{Supp}(F)$  is compact. Let  $\text{D}_{\mathbf{h}}^{\mathbf{b}}(\mathcal{D}^\circ)$  denote the full triangulated subcategory of  $\text{D}^{\mathbf{b}}(\mathcal{D}^\circ)$  consisting of bounded complexes with holonomic cohomology groups. Letting  $\text{DR}(\mathcal{M}) = \mathcal{M} \otimes_{\mathcal{D}}^{\mathbf{L}} \mathcal{O}[-n]$  for an object  $\mathcal{M}$  of  $\text{D}^{\mathbf{b}}(\mathcal{D}^\circ)$ , we have first :

**Lemma 2.1.** *Let  $\mathcal{M}$  be an object of  $\text{D}_{\mathbf{h}}^{\mathbf{b}}(\mathcal{D}^\circ)$ ,  $F$  of  $\text{D}_{\mathbf{R-c}}^{\mathbf{b}}(X)$ . Then there is an isomorphism*

$$(2.1) \quad \text{DR}(\mathcal{M}) \otimes F \cong \text{D}' \text{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes F, \Omega^n),$$

where  $\text{D}' = \text{R}\mathcal{H}om_{\mathbf{C}}(\bullet, \mathbf{C}_X)$ .

The proof will be given later.

Since  $R\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes F, \Omega^n)$  is  $\mathbf{R}$ -constructible, from (2.1), we have

$$D'(\mathrm{DR}(\mathcal{M}) \otimes F) \cong R\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes F, \Omega^n).$$

By the Verdier duality, we get

$$\begin{aligned} R\Gamma(R\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes F, \Omega^n)) [n] &\cong R\mathrm{Hom}_{\mathbf{C}}(R\Gamma_c(\mathrm{DR}(\mathcal{M}) \otimes F), \mathbf{C})[-n] \\ &= R\mathrm{Hom}_{\mathbf{C}}(R\Gamma_c(\mathcal{M} \otimes_D^L \mathcal{O} \otimes F), \mathbf{C}). \end{aligned}$$

This completes the proof of Theorem 1. QED

*Proof of Lemma 2.1.* If  $F = \mathbf{C}_X$ , this duality formula is contained in [KK] and [M2], and we have

$$(2.2) \quad \mathrm{DR}(\mathcal{M}) \cong D' R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \Omega^n).$$

Let  $C = R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \Omega^n)$ ; then, by Kashiwara's theorem [K1],  $C$  is  $\mathbf{C}$ -constructible. Hence

$$D'(D' C \otimes F) \cong R\mathcal{H}om_{\mathbf{C}}(F, C)$$

(see [KS, 3.4.6]), and, since  $D' C \otimes F$  is  $\mathbf{R}$ -constructible,

$$D' C \otimes F \cong D' R\mathcal{H}om_{\mathbf{C}}(F, C).$$

By (2.2), we get

$$\mathrm{DR}(\mathcal{M}) \otimes F \cong D' R\mathcal{H}om_{\mathbf{C}}(F, C).$$

QED

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